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CHAPTER 1. GUIDING PRINCIPLES AND KEY COMPONENTS OF AN EFFECTIVE MATHEMATICS PROGRAM

A long-standing content issue in mathematics concerns the balance between theoretical and applied approaches. Mathematics is both. In the theoretical (pure) sense, mathematics is a subject in its own right with distinct methods and content to be studied. But mathematics is also extremely applicable both in the practical sense and in connection to other realms of study, including the arts, humanities, social sciences, and the sciences. Any comprehensive representation of the content of mathematics must balance these aspects of beauty and power.

—A. Holz, Walking the Tightrope

The major goals of mathematics education can be divided into two categories: goals for teachers and goals for students.

Goals for Teachers

Goals for teachers to achieve are as follows:

1. Increase teachers’ knowledge of mathematics content through professional development focusing on standards-based mathematics.
2. Provide an instructional program that preserves the balance of computational and procedural skills, conceptual understanding, and problem solving.
3. Assess student progress frequently toward the achievement of the mathematics standards and adjust instruction accordingly.
4. Provide the learning in each instructional year that lays the necessary groundwork for success in subsequent grades or subsequent mathematics courses.

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5. Create and maintain a classroom environment that fosters a genuine understanding and confidence in all students that through hard work and sustained effort, they can achieve or exceed the mathematics standards.
6. Offer all students a challenging learning experience that will help to maximize their individual achievement and provide opportunities for students to exceed the standards.
7. Offer alternative instructional suggestions and strategies that address the specific needs of California’s diverse student population.
8. Identify the most successful and efficient approaches within a particular classroom so that learning is maximized.

Goals for Students

Goals for students to achieve are as follows:

1. Develop fluency in basic computational and procedural skills, an understanding of mathematical concepts, and the ability to use mathematical reasoning to solve mathematical problems, including recognizing and solving routine problems readily and finding ways to reach a solution or goal when no routine path is apparent.
2. Communicate precisely about quantities, logical relationships, and unknown values through the use of signs, symbols, models, graphs, and mathematical terms.
3. Develop logical thinking in order to analyze evidence and build arguments to support or refute hypotheses.
4. Make connections among mathematical ideas and between mathematics and other disciplines.

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5. Apply mathematics to everyday life and develop an interest in pursuing advanced studies in mathematics and in a wide array of mathematically related career choices.
6. Develop an appreciation for the beauty and power of mathematics.

It is well known that California students lag behind students in other states and nations in their mastery of mathematics (Reese et al. 1997; Beaton et al. 1996). Comparing the 1990s to the 1970s, a study found that the number of students earning bachelor's and master's degrees in mathematics has decreased during the last 20 years (NCES 1997). At the same time the number of students entering California State University and needing remediation in mathematics has been increasing (California State University 1998). The result of students achieving the goals of this framework and mastering the California mathematics standards will be not only an increase in student mastery of mathematics but also a greater number of students who have the potential and interest to pursue advanced academic learning in mathematics. Because many jobs directly and indirectly require facility with different aspects of applied mathematics (Rivera-Batiz 1992), achieving the goals of this framework will also enable California students to pursue the broadest possible range of career choices.

By meeting the goals of standards-based mathematics, students will achieve greater proficiency in the practical uses of mathematics in everyday life, such as balancing a checkbook, purchasing a car, and understanding the daily news. This process will help the citizens of California understand their world and be productive members of society.

When students delve deeply into mathematics, they gain not only conceptual understanding of mathematical principles but also knowledge of and experience with pure reasoning. One of the most important goals of mathematics is to teach students

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logical reasoning. Mathematical reasoning and conceptual understanding are not separate from content; they are intrinsic to the mathematical discipline that students master at the more advanced levels.

Students who understand the aesthetics and power of mathematics will have a deep understanding of how mathematics enriches their lives. When students experience the satisfaction of mastering a challenging area of human thought, they feel better about themselves (Nicholls 1984). Students who can see the interdependence of mathematics and music, art, architecture, science, philosophy, and other disciplines will become lifelong students of mathematics regardless of the career they pursue.

When students master or exceed the goals of standards-based mathematics instruction, the benefits to both the individual and to society are enormous. Yet achieving these goals is no simple task. Hard work lies ahead. This framework was designed to help educators, families, and communities in California to meet the challenge.

Achieving Balance Within Mathematics—Three Important Components

> At the heart of mathematics is reasoning. One cannot do mathematics without reasoning. . . . Teachers need to provide their students with many opportunities to reason through their solutions, conjectures, and thinking processes. Opportunities in which very young students . . . make distinctions between irrelevant and relevant information or attributes, and justify relationships between sets can contribute to their ability to reason logically.
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> —S. Chapin, The Partners in Change Handbook

Mathematics education must provide students with a balanced instructional program. In such a program students become proficient in basic computational and

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procedural skills, develop conceptual understanding, and become adept at problem solving.

All three components are important; none is to be neglected or underemphasized. Balance, however, does not imply allocating set amounts of time for each of the three components. At some times students might be concentrating on lessons or tasks that focus on one component; at other times the focus may be on two or all three. As described in Chapter 4, “Instructional Strategies,” different types of instruction seem to foster different components of mathematical competence. Nonetheless, recent studies suggest that all three components are interrelated (Geary 1994; Siegler and Stern 1998; Sophian 1997). For example, conceptual understanding provides important constraints on the types of procedures children use to solve mathematics problems; at the same time practicing procedures provides an opportunity to make inductions about the underlying concepts (Siegler and Stern 1998).

Balance Defined

Computational and procedural skills are those that all students should learn to use routinely and automatically. Students should practice basic computational and procedural skills sufficiently and use them frequently enough to commit them to memory. Frequent use is also required to ensure that these skills are retained and maintained over the years.

Mathematics makes sense to students who have a conceptual understanding of the domain. They know not only how to apply skills but also when to apply them and why they are being applied. They see the structure and logic of mathematics and use it flexibly, effectively, and appropriately. In seeing the larger picture and in understanding the underlying concepts, they are in a stronger position to apply their

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knowledge to new situations and problems and to recognize when they have made procedural errors.

Students who do not have a deep understanding of mathematics suspect that it is just a jumble of unrelated procedures and incomprehensible formulas. For example, children who do not understand the basic counting concepts view counting as a rote, a mechanical activity. They believe that the only correct way to count is by starting from left to right and by assigning each item a number (with a number name, such as “one”) in succession (Briars and Siegler 1984). In contrast, children with a good conceptual understanding of counting understand that items can be counted in any order—starting from right to left, skipping around, and so forth—as long as each item is counted only once (Gelman and Meck 1983). A strong conceptual understanding of counting, in turn, provides the foundation for using increasingly sophisticated counting strategies to solve arithmetic problems (Geary, Bow-Thomas, and Yao 1992).

Problem solving in mathematics is a goal-related activity that involves applying skills, understandings, and experiences to resolve new, challenging, or perplexing mathematical situations. Problem solving involves a sequence of activities directed toward a specific mathematical goal, such as solving a word problem, a task that often involves the use of a series of mathematical procedures and a conceptual representation of the problem to be solved (Geary 1994; Siegler and Crowley 1994; Mayer 1985).

When students apply basic computational and procedural skills and understandings to solve new or perplexing problems, their basic skills are strengthened, the challenging problems they encounter can become routine, and their conceptual understanding deepens. They come to see mathematics as a way of finding solutions to problems that occur outside the classroom. Thus, students

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grow in their ability and persistence in problem solving through experience in solving problems at a variety of levels of difficulty and at every level in their mathematical development.

Basic Computational and Procedural Skills

For each level of mathematics, a specific set of basic computational and procedural skills must be learned. For example, students need to memorize the number facts of addition and multiplication of one-digit numbers and their corresponding subtraction and division facts. The ability to retrieve these facts automatically from long-term memory, in turn, makes the solving of more complex problems, such as multistep problems that involve basic arithmetic, quicker and less likely to result in errors (Geary and Widaman 1992). As students progress through elementary school, middle school, and high school, they should become proficient in the following skills:

- Finding correct answers to addition, subtraction, multiplication, and division problems
- Finding equivalencies for fractions, decimals, and percents
- Performing operations with fractions, decimals, and percents
- Measuring
- Finding perimeters and areas of simple figures
- Interpreting graphs encountered in daily life
- Finding the mean and median of a set of data from the real world
- Using scientific notation to represent very large or very small numbers
- Using basic geometry, including the Pythagorean theorem
- Finding the equation of a line, given two points through which it passes
- Solving linear equations and systems of linear equations

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This list, which is by no means exhaustive, is provided for illustrative purposes only. Several factors should be considered in the development and maintenance of basic computational and procedural skills:

- Students must practice skills in order to become proficient. Practice should be varied and should be included both in homework assignments and in classroom activities. Teachers, students, and parents should realize that students must spend substantial time and exert significant effort to learn a skill and to maintain it for the long term (Ericsson, Krampe, and Tesch-Römer 1993).
- Basic computational and procedural skills develop over time, and they increase in depth and complexity through the years. For example, the ability to interpret information presented graphically begins at the primary level and extends to more sophisticated procedures as students progress through the grades.
- The development of basic computational and procedural skills requires that students be able to distinguish among different basic procedures by understanding what the procedures do. Only then will students have the basis for determining when to use the procedures they learn. For example, students must know the procedures involved in adding and multiplying fractions, and they must understand how and why these procedures produce different results.
- To maintain skills, students must use them frequently. Once students have learned to use the Pythagorean theorem, for example, they need to use it again and again in algebraic and geometric problems.
- Students may sometimes learn a skill more readily when they know how it will be used or when they are intrigued by a problem that requires the skill.

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Conceptual Understanding

Conceptual understanding is important at all levels of study. For example, during the elementary grades students should understand that:

- One way of thinking about multiplication is as repeated addition.
- One interpretation of fractions is as parts of a whole.
- Measurement of distances is fundamentally different from measurement of area.
- A larger sample generally provides more reliable information about the probability of an event than does a smaller sample.

As students progress through middle school and high school, they should, for example, understand that:

- The concepts of proportional relationships underlie similarity.
- The level sets of functions of two variables are curves in the coordinate plane.
- Factoring a polynomial function into irreducible factors helps locate the x-intercepts of its graph.
- Proofs are required to establish the truth of mathematical theorems.

Problem Solving

Problems occur in many forms. Some are simple and routine, providing practice for skill development. Others are more complex and take a longer time to complete. Whatever their nature, it is important that the kinds of problems students are asked to solve balance situations in the real world with more abstract situations. The process of solving problems generally has the following stages (Geary 1994; Mayer 1985):

- Formulation, analysis, and translation
- Integration and representation

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- Solutions and justifications

Formulation, analysis, and translation. Problems may be stated in an imprecise form or in descriptions of puzzling or complex situations. The ability to recognize potential mathematical relationships is an important problem-solving technique, as is the identification of basic assumptions made directly or indirectly in the description of the situation, including the identification of extraneous or missing information. Important considerations in the formulation and analysis of any problem situation include determining mathematical hypotheses, making conjectures, recognizing existing patterns, searching for connections to known mathematical structures, and translating the gist of the problem into mathematical representations (e.g., equations).

Integration and representation. Important skills involved in the translation of a mathematical problem into a solvable equation are problems of integration and representation. Integration involves putting together different pieces of information that are presented in complex problems, such as multistep problems. However such problems are represented, a wide variety of basic and technical skills are needed in solving problems; and, given this need, a mathematics program should include a substantial number of ready-to-solve exercises that are designed specifically to develop and reinforce such skills.

Solutions and justifications. Students should have a range of strategies to use in solving problems and should be encouraged to think about all possible procedures that might be used to aid in the solving of any particular problem, including but not limited to the following:

- Referring to and developing graphs, tables, diagrams, and sketches
- Computing
- Finding a simpler related problem

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- Looking for patterns
- Estimating, conjecturing, and verifying
- Working backwards

Once the information in a complex problem has been integrated and translated into a mathematical representation, the student must be skilled at solving the associated equations and verifying the correctness of the solutions. Students might also identify relevant mathematical generalizations and seek connections to similar problems. From the earliest years students should be able to communicate and justify their solutions, starting with informal mathematical reasoning and advancing over the years to more formal mathematical proofs.

Connecting Skills, Conceptual Understanding, and Problem Solving

Basic computational and procedural skills, conceptual understanding, and problem solving form a web of mutually reinforcing elements in the curriculum. Computational and procedural skills are necessary for the actual solution of both simple and complex problems, and the practice of these skills provides a context for learning about the associated concepts and for discovering more sophisticated ways of solving problems (Siegler and Stern 1998). The development of conceptual understanding provides necessary constraints on the types of procedures students use to solve mathematics problems, enables students to detect when they have committed a procedural error, and facilitates the representation and translation phases of problem solving. Similarly, the process of applying skills in varying and increasingly complex problem-solving situations is one of the ways in which students not only refine their skills but also reinforce and strengthen their conceptual understanding and procedural competencies.

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Key Components of an Effective Mathematics Program

Assumption: Proficiency is determined by student performance on valid and reliable measures aligned with the mathematics standards.

In an effective and well-designed mathematics program, students move steadily from what they already know to a mastery of skills, knowledge, and understanding. Their thinking progresses from an ability to explain what they are doing, to an ability to justify how and why they are doing it, to a stage at which they can derive formal proofs. The quality of instruction is a key factor in developing student proficiency in mathematics. In addition, several other factors or program components play an important role. They are discussed in the following section:

I. Assessment

Assessment should be the basis for instruction, and different types of assessment interact with the other components of an effective mathematics program.

In an effective mathematics program:

- Assessment is aligned with and guides instruction. Students are assessed frequently to determine whether they are progressing steadily toward achieving the standards, and the results of this assessment are useful in determining instructional priorities and modifying curriculum and instruction. The assessment looks at the same balance (computational and procedural skills, conceptual understanding, and problem solving) emphasized in instruction.
- Assessments serve different purposes and are designed accordingly. Assessment for determining a student’s placement in a mathematics program should cover a broad range of standards. These broad assessments measure whether or not students have prerequisite knowledge and allow them to demonstrate their full understanding of mathematics. Monitoring student progress

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daily or weekly requires a quick and focused measurement tool. Summative evaluation, which takes place at the end of a series of lessons or a course, provides specific and detailed information about which standards have or have not been achieved.

- Assessments are valid and reliable. A valid assessment measures the specific content it was designed to measure. An assessment instrument is reliable if it is relatively error-free and provides a stable result each time it is administered.
- Assessment can improve instruction when teachers use the results to analyze what students have learned and to identify difficult concepts that need reteaching.

II. Instruction

The quality of instruction is the single most important component of an effective mathematics program. International comparisons show a high correlation between the quality of mathematics instruction and student achievement (Beaton et al. 1996).

In an effective mathematics program:

- Teachers possess an in-depth understanding of the content standards and the mathematics they are expected to teach and continually strive to increase their knowledge of content.
- Teachers are able to select research-based instructional strategies that are appropriate to the instructional goals and to students' needs.
- Teachers effectively organize instruction around goals that are tied to the standards and direct students' mathematical learning.
- Teachers use the results of assessment to guide instruction.

III. Instructional Time

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Study after study has demonstrated the relationship between the time on task and student achievement (Stigler, Lee, and Stevenson 1987, 1283). Priority must be given to the teaching of mathematics, and instructional time must be protected from interruptions.

In an effective mathematics program:

- Adequate time is allocated to mathematics. Every day all students receive at least 50 to 60 minutes of mathematics instruction, not including homework. Additional instructional time is allocated for students who are, for whatever reason, performing substantially below grade level in mathematics. All students are encouraged to take mathematics courses throughout high school.
- Learning time is extended through homework that increases in complexity and duration as students mature. Homework should be valued and reviewed. The purpose of homework is to practice skills previously taught or to have students apply their previously learned knowledge and skills to new problems. It should be assigned in amounts that are grade-level appropriate and, at least in the early grades, it should focus on independent practice and the application of skills already taught. For more advanced students, homework may be used as a means for exploring new concepts.
- During the great majority of allocated time, students are active participants in the instruction. Active can be described as the time during which students are engaged in thinking about mathematics or doing mathematics.
- Instructional time for mathematics is maximized and protected from such interruptions as calls to the office, public address announcements, and extracurricular activities.

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IV. Instructional Resources

All teachers need high-quality instructional resources, but new teachers especially depend on well-designed resources and materials that are aligned with the standards.

In an effective mathematics program:

- Instructional resources focus on the grade-level standards. It may be necessary to go beyond the standards, however, both to provide meaningful enrichment and acceleration for fast learners and to introduce content needed for the mastery of standards in subsequent grades and courses. For example, the Algebra I standards do not mention complex numbers; yet quadratic equations, which often have complex roots, are fully developed in Algebra I. Therefore, an introduction to complex numbers may be included in Algebra I, both to avoid the artificial constraint of having only problems with real roots and to lay the foundation for the mastery of complex numbers in Algebra II.
- Instructional resources are factually and technically accurate and address the content outlined in the standards.
- Instructional resources emphasize depth of coverage. The most critical, highest-priority standards are addressed in the greatest depth. Ample practice is provided.
- Instructional resources are organized in a sequential, logical way. The resources are coordinated from level to level.
- Instructional options for teachers are included. For instance, a teacher’s guide might explain the rationale and procedures for different ways of introducing a topic (e.g., through direct instruction or discovery-oriented instruction) and present various methods for assessing student progress. In addition to providing

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teachers with options, the resources should offer reliable guidelines for exercising those options.

- Resources balance basic computational and procedural skills, conceptual understanding, and problem solving and stress the interdependency of all three.
- Resources provide ample opportunities for students to explain their thinking, verbally and in writing, formally and informally.
- Resources supply ideas or tools for accommodating diverse student performance within any given classroom. They offer suggestions for reteaching a concept, providing additional practice for struggling students, or condensing instruction so that advanced students can concentrate on new material.

V. Instructional Grouping and Scheduling

Research shows that what students are taught has a greater effect on achievement than does how they are grouped (Kulik 1992; Rogers 1991). The first focus of educators should always be on the quality of instruction. Grouping and scheduling are tools that educators can use to improve learning, not goals in and of themselves.

In an effective mathematics program:

- Grouping students according to their instructional needs improves student achievement (Benbow and Stanley 1996). An effective mathematics program (1) uses grouping options in accordance with variability within individual classrooms; and (2) maintains or changes grouping strategies in accordance with student performance on regular assessments.
- Cooperative group work is used judiciously, supplementing and expanding on initial instruction either delivered by teachers or facilitated through supervised exploration. Although students can often learn a great deal from one another and

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can benefit from the opportunity to discuss their thinking, the teacher is the primary leader in a class and maintains an active instructional role during cooperative learning. When cooperative group work is used, it should lead toward students' eventual independent demonstration of mastery of the standards and individual responsibility for learning.

- Cross-grade or cross-class grouping is an alternative to the more arbitrary practice of grouping according to chronological age or grade. Grouping by instructional needs across grade levels increases scheduling challenges for teachers and administrators near the beginning of a school year, but many teachers find the practice liberating later on because it reduces the number of levels for which a teacher must be prepared to teach in a single period.

VI. Classroom Management

Potentially, the primary management tool for teachers is the mathematics curriculum itself. When students are actively engaged in focused, rigorous mathematics, fewer opportunities for inappropriate behavior arise. When students are successful and their successes are made clear to them, they are more likely to become motivated to work on mathematics.

In an effective mathematics program:

- Teachers are positive and optimistic about the prospect that all students can achieve. Research shows that teachers' self-esteem and enthusiasm for the subject matter have a greater effect on student achievement than does students' self-esteem (Clark 1997).
- Classrooms have a strong sense of purpose. Both academic and social expectations are clearly understood by teachers and students alike. Academic expectations relate directly to the standards.

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- Intrinsic motivation is fostered by helping students to develop a deep understanding of mathematics, encouraging them to expend the effort needed to learn, and organizing instruction so that students experience satisfaction when they have mastered a difficult concept or skill. External reward systems are used sparingly; for example, as a temporary motivational device for older students who enter mathematics instruction without the intrinsic motivation to work hard.

VII. Professional Development

The preparation of teachers and support for their continuing professional development are critical to the quality of California schools. Research from other countries suggests that student achievement can improve when teachers are able to spend time together planning and evaluating instruction (Beaton et al. 1996).

In an effective mathematics program:

- Teachers have received excellent preservice training, are knowledgeable about mathematics content, and are able to use a wide variety of instructional strategies.
- Continuing teacher in-service training focuses on (1) enhancing teachers' proficiency in mathematics; and (2) providing pedagogical tools that help teachers to ensure that all students meet or exceed grade-level standards.
- Staff development is a long-term, planned investment strongly supported by the administration and designed to ensure that teachers continue to develop skills and knowledge in mathematics content and instructional options. "One-shot" staff development activities with no relationship to a long-term plan are recognized as having little lasting value.
- As with students, staff development actively engages teachers in mathematics and mathematics instruction. In addition to active involvement during classroom-

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style staff development, teachers have the opportunity to interact with students and staff developers during in-class coaching sessions.

- Individuals who have helped teachers bring their students to high achievement levels in mathematics are called on to demonstrate effective instructional practices with students.
- Teachers are given time and opportunities to work together to plan mathematics instruction. Districts and schools find creative ways to allow time for this planning.

VIII. Administrative Practices

Administrative support for mathematics instruction can help remind all those involved in education that reform efforts are not effective unless they contribute to increased achievement. Administrators can help teachers maintain a focus on high-quality instruction.

In an effective mathematics program:

- Mathematics achievement is among the highest priorities at the school.
- Long-term and short-term goals for the school, each grade level, and individuals are outlined clearly and reviewed frequently.
- Scheduling, grouping, and allocating personnel are shaped by a determination that all students will meet or exceed the mathematics standards.
- Principals demonstrate a strong sense of personal responsibility for achievement within their schools.
- Administrators consider using mathematics specialists to teach most or all of the mathematics classes or to coach other teachers.
- Administrators plan in advance for predictable contingencies, such as the need to realign instructional groups frequently, accommodate students transferring into

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the school, or redesign instruction to include intervention for students performing below grade level.

- Administrators and teachers collaborate on developing schoolwide management systems and schoolwide efforts to showcase mathematics for students, parents, and other members of the community.

IX. Community Involvement

Mathematics education is everybody’s business. Parents, community members, and business and industry can all make significant contributions.

In an effective mathematics program:

- Parents are encouraged to be involved in education and are assisted in supporting their children’s learning in mathematics. Parent comments are encouraged, valued, and used for program planning.
- Materials are organized so that parents, siblings, and community members can provide extended learning experiences.
- The community is used as a classroom that offers abundant examples of how and why mathematics is important in people’s lives, work, and thinking.